Table 1 R_θ and H-values for various C^*-values

$C^* \times 10^{-5}$	Exp. R_θ	Computed R_θ	Computed H
0.8	200	225	1.76
1.0	250	270	1.71
1.2	295	313	1.68
1.8	430	434	1.60
2.4	540	538	1.57
3.0	640	634	1.55
3.7	760	735	1.53

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Comments on Crocco's Solution and the Independence Principle for Compressible Turbulent Boundary Layers

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Nomenclature†

F = $\bar F + F' = \bar u/u_e + u'/u_e$, instantaneous chordwise velocity profile
g = $\bar g + g' = \bar w/w_e + w'/w_e$, instantaneous spanwise velocity profile
h = static enthalpy
H = $h + (u^2+v^2+w^2)/2$, total enthalpy
m = mass flow
M = Mach number
p = pressure
Pr = molecular Prandtl number
q = $(u^2+v^2+w^2)^{1/2}$
R_δ = local Reynolds number based on boundary-layer thickness, $\rho_e u_e \delta/\mu_e$
t = time
u = velocity parallel to surface in x-direction (chordwise component)
v = velocity normal to surface
$\tilde v$ = $\bar v + \overline{\rho' v'}/\rho$
w = velocity parallel to surface in z-direction (spanwise component)
x = distance along surface normal to cylinder generators (chordwise coordinate)
y = distance normal to surface
z = distance along surface parallel to cylinder generators (spanwise coordinate)
γ = ratio of specific heats
δ = boundary-layer thickness
μ = viscosity coefficient
ρ = mass density
θ = $\bar\theta + \theta' = \dfrac{\bar H - H_w}{H_e - H_w} + \dfrac{H'}{H_e - H_w}$, instantaneous total enthalpy profile

Subscripts

e = local freestream
w = wall conditions

Symbols

$(\bar{\ })$ = time mean quantity
$(\ )'$ = fluctuating quantity
$\sim$ = order of magnitude
$\approx$ = approximately equal

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† All local variables without either an overbar or superscript prime are the instantaneous values.

Introduction

IT has been recognized[1] for many years that Crocco's solution[2] to the energy equation for the compressible laminar boundary layer on a flat plate applies to turbulent flow if both the molecular and turbulent Prandtl numbers are unity. Schaubauer and Tchen[3] showed further if the mean flow is exactly parallel, the only requirement for the applicability of the Crocco solution ($\bar\theta = \bar F$) to turbulent flow is that the molecular Prandtl number is unity. Thus, when the limitations of zero streamwise pressure gradient, molecular Prandtl number unity, turbulent Prandtl number unity, and/or parallel flow are nearly satisfied, the Crocco solution provides a useful approximation to the temperature distribution across turbulent boundary layers.[4,5] Another limitation, not usually considered, is that the wall temperature should be constant.[6] Recent results obtained by Feller[7] have shown that the relation between the normalized total temperature and velocity profile parameters is sensitive to the entire history of the wall temperature distribution for hypersonic tunnel-wall boundary layers. Hence, by implication, the Crocco solution is not valid for hypersonic turbulent boundary layers unless both the wall temperature and pressure are nearly constant for essentially the entire history of the flow.

The purpose of the present Note is to show that on a flat plate where both the wall temperature and mean wall pressure are constant, neither of the limitations of parallel flow or of unity for the turbulent Prandtl number are required in order for the Crocco solution to apply to the turbulent boundary-layer flow. It is shown herein that this result is subject to restrictions on the magnitude of pressure fluctuations.

The same analysis is generalized to show that the compressible turbulent boundary layer on an isothermal swept flat plate is independent of the spanwise flow if the molecular Prandtl number is unity. The independence principle,[8–10] which for the present problem may be stated as simply $\bar g = \bar F$,[11] should then apply as a good approximation to the turbulent boundary layer on a swept flat plate for both incompressible flow with arbitrary Prandtl number and to compressible flow with unity Prandtl number. These results are again subject to restrictions on the magnitude of pressure fluctuation terms.

Analysis

To define the problem, the Reynolds averaged momentum and energy equations for turbulent boundary-layer flow on an infinite swept cylinder[10] are written in terms of the normalized mean profile variables $\bar F$, $\bar g$, and $\bar\theta$[12] and their corresponding fluctuating components.

Chordwise mean momentum equation

$$\bar\rho \bar F \frac{\partial \bar F}{\partial x} + \frac{\bar\rho \tilde v}{u_e}\frac{\partial \bar F}{\partial y} = -\frac{1}{u_e^2}\frac{\partial \bar p}{\partial x} - \frac{\bar\rho \bar F^2}{u_e}\frac{du_e}{dx} + \frac{1}{u_e}\frac{\partial}{\partial y}\left(\bar\mu \frac{\partial \bar F}{\partial y} - \overline{(\rho v)' F'}\right) \tag{1}$$

Spanwise mean momentum equation

$$\bar\rho \bar F \frac{\partial \bar g}{\partial x} + \frac{\bar\rho \bar v}{u_e}\frac{\partial \bar g}{\partial y} = \frac{1}{u_e}\frac{\partial}{\partial y}\left(\bar\mu \frac{\partial \bar g}{\partial y} - \overline{(\rho v)' g'}\right) \tag{2}$$

Total enthalpy equation

$$\bar\rho\bar F\frac{\partial\bar\theta}{\partial x} + \frac{\bar\rho\bar v}{u_e}\frac{\partial\bar\theta}{\partial y} = -\frac{\bar\rho\bar F(1-\bar\theta)}{H_e-H_w}\frac{dH_w}{dx} + \frac{1}{u_e}\frac{\partial}{\partial y}\left\{\frac{\bar\mu}{Pr}\left[\frac{\partial\bar\theta}{\partial y}+\frac{Pr-1}{2(H_e-H_w)}\left(u_e^2\frac{\partial \bar F^2}{\partial y}+w_e^2\frac{\partial\bar g^2}{\partial y}\right)\right]-\overline{(\rho v)'\theta'}\right\} \tag{3}$$

The boundary conditions at the wall ($y = 0$) and the freestream ($y = y_e \to \infty$) for the dependent mean profile variables are identical as follows:

$$\left.\begin{aligned}\bar F(x,0) &= \bar g(x,0) = \bar\theta(x,0) = 0\\ \bar F(x,y_e) &= \bar g(x,y_e) = \bar\theta(x,y_e) = 1.0\end{aligned}\right\} \tag{4}$$

For the present purposes, the boundary conditions on the fluctuating components of these variables may be specified as

$$\left.\begin{aligned}F'(x,0) &= g'(x,0) = \theta'(x,0) = 0\\ F'(x,y_e) &= g'(x,y_e) = \theta'(x,y_e) = 0\end{aligned}\right\} \tag{5}$$

Thus, it can be seen from Eqs. (1–3) that if the following conditions are satisfied

$$\frac{\partial \bar p}{\partial x} = 0,\quad \frac{du_e}{dx} = 0,\quad \frac{dH_w}{dx} = \frac{dh_w}{dx} = 0,\quad H_w \neq H_e \tag{6}$$

$$Pr = 1.0 \tag{7}$$

$$\overline{(\rho v)'F'} = \overline{(\rho v)'g'} = \overline{(\rho v)'\theta'} \tag{8}$$

then $\bar g = \bar F$ and $\bar\theta = \bar F$ become solutions to Eqs. (2) and (3). If $\bar\theta = \bar F$ (the Crocco integral) and $\theta' = F'$[13] can be admitted as solutions to Eq. (3), Morkovin's[14] concept of a "Strong Reynolds Analogy" follows by evaluating the ratio $(\partial\bar H/\partial y)/(\partial\bar u/\partial y)$ at the wall. In the following analysis, some general limitations on applications of Morkovin's concept will become apparent.

The conditions of Eq. (6) are satisfied by a cooled or heated isothermal flat plate (for subsonic flow, the plate would have to be at zero incidence). Condition (7) is only approximately satisfied by most gases; however, the contribution of the pertinent viscous term in Eq. (3) is small in turbulent flows except very near the wall. To determine when Eqs. (8) are applicable, it is necessary to consider the governing equations for the instantaneous variables F, g, and θ, since by multiplying these variables by $(\rho v)'$ and averaging the results, it is evident that Eqs. (8) are valid solutions only when $F = g = \theta$.

The governing equations for the instantaneous variables in a thin compressible turbulent boundary layer on a swept infinite cylinder where all flow quantities outside the boundary layer are time steady may be written as (the numbers above the terms are order of magnitude estimates which will be explained in the following paragraph)

Continuity equation

$$\overset{0.1}{\frac{\partial\rho'}{\partial t}} + \overset{0.1}{\rho\frac{\partial u}{\partial x}} + \overset{0.1}{u\frac{\partial\rho}{\partial x}} + \overset{0.1}{\rho\frac{\partial v}{\partial y}} + \overset{0.1}{v\frac{\partial\rho}{\partial y}} + \overset{0.1}{\rho\frac{\partial w'}{\partial z}} + \overset{0.1}{w\frac{\partial\rho'}{\partial z}} = 0 \tag{9}$$

x-momentum equation

$$\overset{0.1}{\rho\frac{\partial u'}{\partial t}} + \overset{0.1}{\rho u\frac{\partial u}{\partial x}} + \overset{0.1}{\rho v\frac{\partial u}{\partial y}} + \overset{0.1}{\rho w\frac{\partial u'}{\partial z}} = -\frac{\partial p}{\partial x} + \overset{1/R_\delta}{\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)} + \text{higher order viscous terms} \tag{10}$$

z-momentum equation

$$\overset{0.1}{\rho\frac{\partial w'}{\partial t}} + \overset{0.1}{\rho u\frac{\partial w}{\partial x}} + \overset{0.1}{\rho v\frac{\partial w}{\partial y}} + \overset{0.1}{\rho w\frac{\partial w'}{\partial z}} = -\frac{\partial p'}{\partial z} + \overset{1/R_\delta}{\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)} + \text{higher order viscous terms} \tag{11}$$

Total enthalpy equation

$$\overset{0.1}{\rho\frac{\partial\theta'}{\partial t}} + \overset{0.1}{\rho u\frac{\partial\theta}{\partial x}} + \overset{0.1}{\rho v\frac{\partial\theta}{\partial y}} + \overset{0.1}{\rho w\frac{\partial\theta'}{\partial z}} = \frac{\partial p'}{\partial t}\Big/(H_e-H_w) + \overset{1/R_\delta}{\frac{\partial}{\partial y}\left\{\frac{\mu}{Pr}\left[\frac{\partial\theta}{\partial y}+\frac{Pr-1}{2(H_e-H_w)}\frac{\partial q^2}{\partial y}\right]\right\}} + \text{higher order viscous terms} \tag{12}$$

To simplify order of magnitude estimates of the enthalpy terms in Eq. (12), the normalized enthalpy profile θ has been introduced. Also, H_w has been assumed constant. From Eqs. (10–12), together with the definitions of the normalized instantaneous variables (see nomenclature) and their boundary conditions, it now becomes apparent that

$$\left.\begin{aligned}g &= F,\quad g' = F'\\ \text{and}\qquad \theta &= F,\quad \theta' = F'\end{aligned}\right\} \tag{13}$$

are valid approximate solution to Eqs. (11) and (12) when u_e, w_e, H_e, and H_w are constants, when $Pr = 1.0$, and when the pressure derivative terms are negligible compared to dominant terms.

To determine when this latter condition would apply, we examine a small portion of the boundary layer of order δ in extent in the x and z directions. To obtain order of magnitude estimates, all velocities are normalized with the local external velocity $q_e \approx (u_e^2+w_e^2)^{1/2}$, all lengths with the local boundary-layer thickness δ, and density with the local ρ_e. Then, if the turbulent eddies or disturbances have scales the order of δ and if fluctuations in velocity, density, and enthalpy are taken as the order of 0.1 of their external values,[15] the resulting order of magnitudes of all terms in Eqs. (9–12), except the pressure terms, are indicated above each term. The values given apply best to the outer part of the boundary layer, say for $y/\delta > 0.2$. For $y/\delta < 0.2$, the turbulence scales are smaller than δ, and the order of magnitude for some of the terms would increase to 1. The maximum change in density is in the y-direction and is here taken as ~ 1 which then approximates adiabatic wall conditions at hypersonic freestream velocities.

When the derivative of the mean pressure $\bar p$ with respect to x is zero, the normalized pressure derivative terms in Eqs. (10–12) become

$$\frac{\partial}{\partial x/\delta}\left(\frac{p'}{\rho_e q_e^2}\right),\quad \frac{\partial}{\partial z/\delta}\left(\frac{p'}{\rho_e q_e^2}\right),\quad \frac{\delta}{u_e}\frac{\partial}{\partial t}\left[\frac{p'}{\rho_e(H_e-H_w)}\right] \tag{14}$$

Since $x/\delta \sim z/\delta \sim tu_e/\delta \sim 1.0$ and since for supersonic speeds $H_e \sim q_e^2$, these derivatives are all of the same order ($H_e = H_w$ is again excluded). From semiempirical analyses[16] (applicable primarily to low-speed flows), a major source of the pressure fluctuations arises near the edge of the viscous sublayer, and the pressure fluctuations at the wall are larger than at any point in the boundary layer. Thus, from correlations[15] of data for p_w' all of these pressure derivative terms would be, at most, ~ 0.001 for $M_e > 4$. This order-of-magnitude estimate is valid even for $M_e \approx 20$ as indicated by results of Fischer et al.[17] where rms mass flow fluctuations as large as 50% were measured in the inner portion of a turbulent boundary layer in helium. If it is assumed[17] that in the outer part of the boundary layer [where $(\overline{m'^2}/\bar m)^{1/2} < 40\%$] the mass flow fluctuations are composed mostly of pressure fluctuations, it follows that for the conditions of Ref. 17

$$p'/\bar p \approx \gamma m'/\bar m \approx 0.7$$

Thus, even though this estimated value is larger than the measured wall value $p_w'/p_e \approx 0.2$ (Ref. 17), the normalized value appropriate for the present purpose, at very large Mach number becomes

$$p'/\rho_e u_e^2 = p'/\gamma p_e M_e^2 \sim 0.001$$

For $M_e < 4$, $(p_w'/\rho_e q_e^2) \sim 0.003$[15] which would rapidly decrease to ~ 0.001 or less, a small distance from the wall.[16] By comparison with the order of magnitude of the other terms in Eqs. (10–12) it then becomes apparent that these pressure derivative terms can be neglected in the instantaneous equations of motion for a thin boundary layer except perhaps for low Mach number conditions near the wall. Since $R_\delta > 10^3$ for typical turbulent boundary layers,[15] the viscous terms are also negligible, but would have to be retained to insure the correct boundary conditions on velocities and enthalpy. Of course, very close to the wall, u, v, and $w \to 0$ as $y \to 0$, while the pressure derivative terms remain finite. Thus, to compute the details of the instantaneous motion very close to the wall, the pressure derivate terms in Eqs. (10–12) would also be required. These derivatives of the fluctuating pressure would balance fluctuations in the viscous terms. Subject to these restrictions on the pressure fluctuations, Eqs. (13) are then solutions to the instantaneous equations of motion when u_e, w_e, H_e, and H_w are constants and when $Pr = 1.0$.

Conclusions

The only restrictions on the application of Crocco's solution ($\bar{\theta} = \bar{F}$) and the independence principle ($\bar{g} = \bar{F}$) to boundary-layer flows, besides those just considered for the pressure fluctuations, are that u_e, w_e, and H_w must all be constant, $\bar{p} = \bar{p}(y)$ only, and $Pr = 1.0$. Since Eqs. (8) are now also applicable, an eddy diffusivity formulation for these conditions would require that the "total" turbulent Prandtl number[6] be unity and also that the eddy viscosity for the shear in the x and z directions be identical. It is also of interest to note that the "invariant turbulence" concept[10] used to model the eddy viscosity for swept cylinder flows[12] reduces to the correct form, consistent with the present analysis, for a swept flat plate. Morkovin's "Strong Reynolds Analogy" concept which was obtained[14] by extending the solutions $\bar{\theta} = \bar{F}$ and $\theta' = F'$ to the surface, can now be expected to apply better for $M > 4$ than at lower Mach numbers where the pressure fluctuation terms may become significant for $y \to 0$. Recent experimental data[18,19] tend to confirm this expectation for the ratio of surface heat transfer to skin friction. One can speculate further that this and other consequences of the Strong Reynolds Analogy may apply better in free flight than in noisy wind tunnels where facility generated noise may become nearly as large as p_w' measured under turbulent boundary layers.[20–22] However, Morkovin's application of this concept to adiabatic flows, wherein $H' = O$ (Ref. 14), is excluded in the present analysis by the requirement that $H_w \neq H_e$.

By use of the same analysis as given previously, it can be easily shown, subject to the restrictions just enumerated, that the Crocco solution applies to two-dimensional or quasi-two-dimensional flows including axisymmetric flows. The restriction of $Pr = 1.0$ is not required for the independence principle to apply to incompressible flow on a flat plate since, for constant density, the energy Eq. (3) is uncoupled from the momentum Eqs. (1) and (2). However, for incompressible flow, the pressure fluctuation terms Eq. (14) may not be negligible close to the wall. The experimental results of Ashkenas and Riddell[9] seem to favor this possibility that the independence principle does not apply accurately to the incompressible turbulent boundary layer on a yawed flat plate, especially in the region close to the wall.

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Finite Twisting and Bending of Thin Orthotropic Strip of Lenticular Section

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THE large-deflection solution for a thin strip of lenticular parabolic section made of isotropic material when subjected to combined moment M and torque T was obtained by

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